PHYSICAL REVIEW E VOLUME 53, NUMBER 4 APRIL 1996

Nonlinear electrohydrodynamic waves on films falling down an inclined plane

A. González[1,2] and A. Castellanos[1]
[1]Departamento de Electrónica y Electromagnetismo, Universidad de Sevilla, Avenida Reina Mercedes, s/n, 41012 Sevilla, Spain
[2]Departamento de Física Aplicada, Universidad de Sevilla, Avenida Reina Mercedes, s/n, 41012 Sevilla, Spain

(Received 10 July 1995)

The stability of a perfectly conducting viscous film, falling down an inclined plane, is considered for the case of an applied uniform normal electric field. A highly nonlinear evolution equation for the deformation of the free surface of the film is derived. The study of the linear stability of the system shows the destabilizing effect of the electric forces. A weakly nonlinear analysis leads to a Ginsburg-Landau equation, which predicts that the destabilization induced by the electric field in an otherwise stable film occurs in the form of traveling waves of finite amplitude.

PACS number(s): 47.65.+a

I. INTRODUCTION

The wave motion in a thin film may often be observed in everyday life. A great deal has been learned about the onset of film waves and their weakly nonlinear evolution, but their often observed strong nonlinear character remains to be quantitatively understood.

In this paper we take an analytical approach to the problem of film waves on an inclined plane when the film is subjected to an electric field uniform at infinity. The nonelectrical problem has been widely studied in recent years. Reviews of the topic have been done by Chang [1] and Lin [2]. Benjamin [3] and Yih [4] solved the linear stability problem for the basic flow of constant thickness and determined the critical Reynolds number for instability. The highly nonlinear character of the phenomena has motivated several approaches: perturbative expansions [5,6], normal form analysis [7,8], and boundary layer theories [9]. The theoretical results, however, do not explain completely the experimental [10–12] and numerical [13] results.

The dynamics of film waves has received much attention from various industries because it has a dramatic effect on transport rates of mass, heat, and momentum. The interest in considering the electric field is due to its possible application to industrial processes, such as the enhancement of heat transfer. The electric field has been used in these devices to destabilize the liquid films on the walls.

II. DESCRIPTION OF THE SYSTEM

We consider a thin film of a viscous liquid falling down an inclined plane. The film has thickness h and the angle between the plane and the horizontal is β. The mass density of the liquid is ρ and its kinematic viscosity is ν and it is assumed to be a perfect electric conductor. The surface tension between the liquid and the surrounding air is given by σ. On the surface, an electric field is applied that at infinity is uniform and perpendicular to the unperturbed surface (Fig. 1).

The equations describing the system are the Navier-Stokes equations for the liquid and the Laplace equation for the electric potential in the air. There is a set of boundary conditions expressing the fact that the liquid is attached to the wall and, at the same time, is electrically grounded. The imposed electric field acts upon the electric charge that lies on the surface.

For the system of equations, there exists a basic solution analogous to the solution for the nonelectric case, which was discovered by Nusselt. In this solution all the magnitudes are independent of the longitudinal coordinate x and the time t. This solution, depending only on the transverse coordinate z, is given by

$$u_0 = \frac{g\sin\beta}{2\nu}(2h_0 z - z^2)\ , \tag{1a}$$

$$w_0 = 0\ , \tag{1b}$$

$$p_0 = p_{\rm atm} - \frac{\varepsilon_0 E_0^2}{2} - \rho g\cos\beta(z-h_0)\,, \tag{1c}$$

$$\phi_0 = E_0(z-h_0)\,, \tag{1d}$$

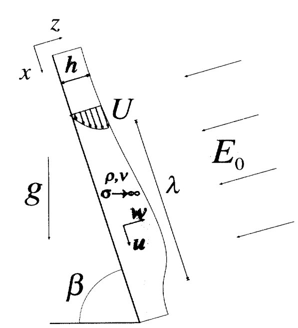

FIG. 1. The system under consideration.

1063-651X/96/53(4)/3573(6)/$10.00

where the layer width h_0 is assumed to be uniform. The steady velocity profile is semiparabolic, while the electric potential varies linearly (see Fig. 1).

III. SCALES AND PARAMETERS

In the following, we will consider changes to these quantities. In order to make the result universal, we must consider appropriate scales for the perturbed quantities.

We will take as the scale for the longitudinal component x a quantity λ that characterizes the typical length of the surface deformation. We will consider that λ is much greater than the depth h_0 (long-wave limit). The latter distance will be taken as the scale for the transverse component z in the liquid. This scale, however, is not appropriate for measuring distances in the air that extend to infinity. Instead, we will scale z in the air with the unit λ, since potential theory shows that perturbations in the electric field extend up to a distance similar to the extent of the sources, in this case the changes in the free surface charge density.

For time, the mechanical period λ/U_0 will be used, where U_0 is the base velocity at the unperturbed surface,

$$U_0 = \frac{g \sin\beta h_0^2}{2\nu} \ . \tag{2}$$

The velocity components will be scaled with the ratio of the typical length to the time scale,

$$u_0 = \frac{\lambda}{T} = U_0 \,, \quad w_0 = \frac{h_0}{T} = \frac{h_0}{\lambda} U_0 \,. \tag{3}$$

The unit for changes in the pressure is taken from the Bernouilli's law as $p_0 = \rho U_0^2$. Finally, the unit for the electric potential is taken from the change in the basic potential, $\phi_0 = E_0 h_0$.

This set of scales gives us the following system of nondimensional equations.

(1) Equations in the liquid layer ($0 < z < h$):

$$u_x + w_z = 0 \ , \tag{4a}$$

$$u_t + (U+u)u_x + w(U_z + u_z) = -p_x + \frac{1}{\delta R}(u_{zz} + \delta^2 u_{xx}) \ , \tag{4b}$$

$$w_t + (U+u)w_x + w w_z = -\frac{1}{\delta^2} p_z + \frac{1}{\delta R}(w_{zz} + \delta^2 w_{xx}) \ . \tag{4c}$$

(2) Equations in the air ($\delta h < \zeta < \infty$):

$$\phi_{xx} + \phi_{\zeta\zeta} = 0 \ . \tag{5}$$

(3) Boundary conditions at the wall ($z = 0$):

$$u = 0 \,, \qquad w = 0 \,. \tag{6}$$

(4) Boundary conditions at the surface ($z = h$, $\zeta = \delta h$):

$$w = h_t + (U+u)h_x \,, \tag{7a}$$

$$\phi - h + 1 = 0 \ , \tag{7b}$$

$$(U_z + u_z + \delta^2 w_x) + 4\delta^2 h_x w_z = 0 \ , \tag{7c}$$

$$\frac{W_e}{2}\left[(1+\delta\phi_\zeta)^2(1+\delta^2 h_x^2) - 1\right] - \delta w_z \frac{1+\delta^2 h_x^2}{1-\delta^2 h_x^2} + \frac{R}{2}p - \cot\beta(h-1) = -\frac{W\delta^2 h_{xx}}{(1+\delta^2 h_x^2)^{3/2}} \ . \tag{7d}$$

(5) Boundary conditions at infinity:

$$\phi_z \to 0 \ . \tag{8}$$

In the above set of equations, U means the dimensionless base velocity $U = 2z - z^2$ and ζ is the transverse component in the air $\zeta = z\delta$. Since the magnitudes depending on z or ζ only mix in the boundary conditions where both coordinates have a definite value, there is no problem in using both variables. The dimensionless parameters are the relative depth δ, the Reynolds number R, the Weber number W, and an electric parameter, W_e, which may be called the "electric Weber number."

$$\delta = \frac{h_0}{\lambda} \,, \quad R = \frac{U_0 h}{\nu} = \frac{g \sin\beta h_0^3}{2\nu^2} \,, \tag{9a}$$

$$W = \frac{T}{\rho g h_0^2 \sin\beta} \,, \qquad W_e = \frac{\varepsilon_0 E_0^2}{\rho g h_0 \sin\beta} \ . \tag{9b}$$

Since we will consider the long-wave limit, we assume that the relative depth $\delta = h_0/\lambda$ is much smaller than unity. The essential role played by the surface tension in the formation and stabilization of the film waves is well known [13]. Therefore, we can also expect that the electric field will have an appreciable effect when its corresponding parameter is comparable to the capillary one. Both effects for the perturbed surface appear in Eq. (7d) preceded by the factors $W\delta^2$ and $W_e\delta$, respectively. These terms will be comparable whenever

$$\frac{W_e}{W\delta} = \frac{\varepsilon_0 E_0^2 \lambda}{\sigma} \simeq 1 \ . \tag{10}$$

This happens to be the case for a conducting thin film ($h \simeq 1\,\mathrm{mm}$) of water or glycerin ($\sigma \simeq 0.07$ N/m) subjected to a normal electric field of the order of 1 MV/m. This field, although strong, is still only one third of the dielectric breakdown field in air. On the other hand, for mercury films ($\sigma \simeq 0.44$ N/m) the required electric field would exceed the dielectric breakdown field and it would be neccesary to replace the air with a vacuum, inert gases, or an insulating liquid.

Kapitza [10] considered the strong surface tension case making $W = O(\delta^{-2})$. In this paper, we assume that the electric field has a comparable effect. Hence, $W_e = O(\delta^{-1})$.

We will study successive approximations to the complete system, retaining greater powers in δ each time.

IV. SOLUTION AT ZEROTH ORDER

To find a solution of the system we must impose a solvability condition on the magnitudes. This condition may come from the integral mass conservation law

$$h_t + Q_x = 0 \ , \qquad \text{where} \qquad Q = \int_0^h (U+u)\,dz. \quad (11)$$

Expanding this equation up to the lowest order in powers of δ, we arrive at the equation

$$h_t + 2h^2 h_x = 0 \ . \quad (12)$$

This equation is also found in the nonelectrical case [14,6,5] and represents forward breaking waves. For a given deformation, its profile steepens at the front and the parameter δ becomes locally finite and higher-order terms are needed. In the limit of small amplitude, the velocity of the linear waves is twice the speed of the basic flow at the surface.

Assuming that the solvability condition is verified, we can obtain the magnitudes up to the lowest order. These are

$$u = 2(h-1)z \ , \quad (13a)$$

$$w = -h_x z^2 \ , \quad (13b)$$

$$p = \frac{2}{R}[-W\delta^2 h_{xx} + \cot\beta(h-1) + W_e\delta(\mathcal{H}(h-1))_x] \ , \quad (13c)$$

$$\phi = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{z(h(x')-1)}{z^2+(x-x')^2}dx' \ , \quad (13d)$$

where $\mathcal{H}(h)$ means the Hilbert transform, defined by

$$\mathcal{H}(h) \equiv \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}dx'\frac{h(x')}{x'-x} \ , \quad (14)$$

with $\mathcal{P}$ the principal value of the integral.

V. SOLVABILITY CONDITION AT FIRST ORDER

If we retain up to the first order in δ and substitute in the terms of this order the above results, we can integrate the system and obtain a solvability condition for the system, given as an equation for h:

$$h_t + [\tfrac{2}{3}h^3 + \delta(\tfrac{8}{15}Rh^6 - \tfrac{2}{3}\cot\beta h^3)h_x + \tfrac{2}{3}W\delta^2 h^3 h_{xxx} - \tfrac{2}{3}\delta W_e h^3(\mathcal{H}(h-1))_{xx}]_x = 0 \ . \quad (15)$$

This equation differs from that found by Nakaya [5] and Lin [6] only in the new term due to the electric field.

A. Linear analysis

To study the effect of the new term, we can consider the stability of the basic flow against infinitesimal perturbations. If we assume that the surface deformation is near the basic state, i.e., $h = 1+\eta$, with $\eta \ll 1$, we get to the linear equation

$$\eta_t + 2\eta_x + \delta[(\tfrac{8}{15}R - \tfrac{2}{3}\cot\beta)\eta_{xx} + \tfrac{2}{3}W\delta^2\eta_{xxxx} - \tfrac{2}{3}W_e\delta(\mathcal{H}(\eta))_{xxx}] = 0 \ . \quad (16)$$

Considering solutions of the form $\eta = A\exp(st+ikx)$ we arrive at the dispersion relation (cf. [7])

$$s + 2ik - (\tfrac{8}{15}R - \tfrac{2}{3}\cot\beta)k^2 + \tfrac{2}{3}Wk^4 - \tfrac{2}{3}W_e k^2|k| = 0 \ . \quad (17)$$

Splitting this relation into its real and imaginary parts, we get

$$\mathrm{Re}\, s = \delta[(\tfrac{8}{15}R - \tfrac{2}{3}\cot\beta)k^2 - \tfrac{2}{3}Wk^4 + \tfrac{2}{3}W_e k^2|k|] \ , \quad (18a)$$

$$\mathrm{Im}\, s = -2k \ . \quad (18b)$$

We can see that the electric term is a destabilizing one. We have several cases.

(1) If $R > R_c = 5\cot\beta/4$, there exists, as in the nonelectrical case, a band of unstable wavelengths, which extends from $k=0$. The most unstable mode has a wave number

$$k_u = \frac{3W_e}{8\delta W} + \frac{1}{8\delta W}\sqrt{9W_e^2 - 32WD} \ , \quad (19)$$

with $D = \cot\beta - 4R/5$. This wave number reduces to $k_u = (|D|/2W\delta^2)^{1/2}$ in the absence of an electric field.

(2) If $R < R_c$, the system is always stable if there is no electric field. If such a field is imposed, we have two more cases.

(a) If $W_e < 2(DW)^{1/2}$, all modes are stable. The curve for the growth rate is a polynomial one, which never crosses the $Re(s) = 0$ axis.

(b) If $W_e > 2(WD)^{1/2}$, there appears a band of unstable modes around k_u (see Fig. 2).

In what follows we will consider the changes in the system behavior for $R < R_c$, when W_e goes above or below its critical value.

B. Bifurcation analysis

From the linear equation (16) we can conclude that below the critical value $W_{ec} = 2(WD)^{1/2}$ all modes are stable, whereas above this value there appears a band of unstable modes around the most unstable one, namely,

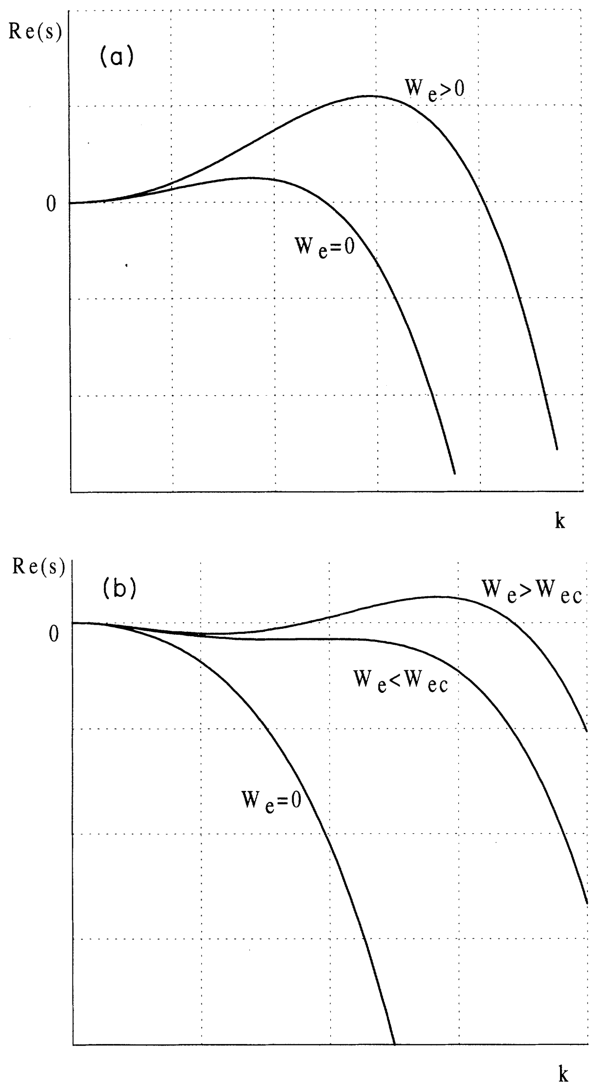

FIG. 2. Changes in the dispersion relation due to the electric field, for a Reynolds number above (a) or below (b) the critical one.

$k_c = (D/2W\delta^2)^{1/2}$. For a value of W_e slightly above the critical value, $W_e = W_{ec}(1+\mu)$, μ being small, the width of the unstable band is of order $\mu^{1/2}$. To study the behavior of these modes we have to introduce nonlinear terms from (15). Putting $h = 1+\eta$, with $\eta \ll 1$ and retaining up to the first order in η^2 and $\delta\eta$ we get

$$\eta_t + 2\eta_x + 4\eta\eta_x + \frac{2\delta}{3}[-D\eta_{xx} + W\delta^2\eta_{xxxx} - W_e\delta(\mathcal{H}(\eta))_{xxx}] = 0. \quad (20)$$

This equation is a modified Kuramoto-Sivashinsky equation with a new integral term. This term makes this equation similar to the Benjamin-Ono equation.

To study (20) we can take away the traveling wave term $2\eta_x$ using a moving frame with the linear velocity. We then use the coordinates defined by

$$\tilde{x} = x - 2t \ , \quad t = t \ . \quad (21)$$

We can also simplify the equation through a rescaling of the magnitudes around the critical point. With

$$\tilde{x} = \delta\left(\frac{W}{D}\right)^{1/2} x^* \ , \quad t = \frac{3\delta W}{2D^2}t^* \ , \quad \eta = \frac{D^{3/2}}{6W^{1/2}}\eta^* \ , \quad (22)$$

$$W_e = 2(WD)^{1/2}(1+\mu) \ . \quad (23)$$

The new parameter μ will be the bifurcation parameter. With this rescaling the equations become (dropping the asterisks)

$$\eta_t + \eta\eta_x - \eta_{xx} + \eta_{xxxx} - 2(1+\mu)\mathcal{H}(\eta)_{xxx} = 0. \quad (24)$$

For a small enough value of μ only a thin band of unstable modes around a central one appears. We can then argue that the main features of the system behavior can be obtained from an analysis of these modes.

This analysis follows essentially the study developed by Manneville in [15]. We consider a wave packet around the central mode and obtain an equation for the envelope from a multiple-scale expansion.

Consider first a single mode of wave number k,

$$\eta(x,t) = A \sim A_k(0)e^{st+ikx} \ . \quad (25)$$

We obtain, after substitution in (24), that the growth factor s is of order $O(\mu)$ but $k \sim 1 + O\mu^{1/2}$. In terms of a parameter $\epsilon = O(\mu^{1/2})$, we have

$$\eta(x,t) = A_k(t) \sim \left(A_k(0)e^{a\epsilon^2 t + i\epsilon x}\right)e^{ix} = A(X,T)e^{ix} \ , \quad (26)$$

where $T = \epsilon^2 t$, $X = \epsilon x$. This suggests a multiple-scale analysis with

$$T = \epsilon^2 t \ , \quad x = x \ , \quad X = \epsilon x \ . \quad (27)$$

We will also expand the surface deformation $\eta(x,X,T)$ and the bifurcation parameter as

$$\eta = \sum_{i=1}^{\infty} \eta^{(i)}\epsilon^i \ , \quad \mu = \sum_{i=2}^{\infty} \mu^i \epsilon^i \ . \quad (28)$$

Note that the expansion of η begins with ϵ, whereas the expansion of μ does it with ϵ^2.

1. Lowest order

We expand Eq. (24) in powers of the small parameter ϵ and obtain, at the lowest order, the equation

$$-\eta^{(1)}_{xx} + \eta^{(1)}_{xxxx} - 2\mathcal{H}(\eta^{(1)})_{xxx} = 0 \ . \quad (29)$$

This is a linear equation whose solution is a combination of exponentials

$$\eta^{(1)} = A_{10} + A_{11}e^{ix} + A_{11}^{*}e^{-ix} , \quad (30)$$

where A^* means the complex conjugate of A. The coefficient A_{10}, however, must be null, because of the volume conservation.

2. Second order

At the following order, we have

$$-\eta_{xx}^{(2)} + \eta_{xxxx}^{(2)} - 2\mathcal{H}(\eta^{(2)})_{xxx}$$
$$= -\eta^{(1)}\eta_x^{(1)} + 2\eta_{xX}^{(1)} - 4\eta_{xxxX}^{(1)} + 6\mathcal{H}(\eta^{(1)})_{xxX} . \quad (31)$$

This condition does not impose any restriction on the long scale X, because the corresponding terms cancel each other. Instead, we have a linear equation with a quadratic forcing term. The solution for $\eta^{(2)}$ is a combination of a particular solution and a solution of the homogenous equation,

$$\eta^{(2)} = A_{20} + A_{21}e^{ix} + A_{22}e^{2ix} + \text{c.c.} \quad (32)$$

As before, the coefficient A_{20} must vanish. The amplitude of the forced response is given by

$$A_{22} = -\frac{i}{4}A_{11}^2 . \quad (33)$$

This result means that the first effect of the unstable modes is a coupling with themselves. Since the coupling is quadratic, it excites a band of modes around $2k_c$ that were initially stable. The evolution of these secondary modes is fully dominated by the primary modes. A further coupling between the excited and the unstable modes leads to a resonant term of wave number k_c. The expected result will be a cubic term in the evolution equation for A_{11}.

3. Third order

To this order the instability becomes explicit. We have the equation

$$-\eta_{xx}^{(3)} + \eta_{xxxx}^{(3)} - 2\mathcal{H}(\eta^{(3)})_{xxx}$$
$$\begin{aligned} &= -\eta^{(2)}\eta_x^{(1)} - \eta^{(1)}\eta_x^{(2)} + 2\eta_{xX}^{(2)} \\ &\quad -4\eta_{xxxX}^{(2)} + 6\mathcal{H}(\eta^{(2)})_{xxX} - \eta_T^{(1)} + \eta_{XX}^{(1)} \\ &\quad -6\eta_{xxXX}^{(1)} + 6\mathcal{H}(\eta^{(1)})_{xXX} \\ &\quad +\mu^{(2)}\mathcal{H}(\eta^{(1)})_{xxx} . \end{aligned} \quad (34)$$

To avoid the appearance of secular terms we have to impose that the coefficient of $\exp(ix)$ on the right hand side vanishes (Fredholm's Alternative). This gives us a new equation for the amplitude of the envelope,

$$A_{11T} - \mu^{(2)}A_{11} + \tfrac{1}{4}|A_{11}|^2A_{11} - A_{11XX} = 0 . \quad (35)$$

This a Ginsburg-Landau equation. It can be encountered in a variety of problems and has been widely studied.

VI. ANALYSIS OF THE ENVELOPE EQUATION

Renaming A_{11} as A we can write Eq. (35) as

$$A_T - \mu A + \tfrac{1}{4}|A|^2A - A_{XX} = 0 . \quad (36)$$

We can assume, as a first approach, that the amplitude does not depend on the spatial variable X. This is almost true when the band of unstable modes is narrow. In this case (36) reduces to a Stuart-Landau equation,

$$A_{0T} - \mu A_0 + \frac{1}{4}|A_0|^2A_0 = 0 . \quad (37)$$

The behavior predicted by this equation depends on the sign of μ. For $\mu < 0$ we have only a stationary amplitude, namely, $A = 0$, which is stable. For $\mu > 0$ we have three stationary values of A, $A = 0$ and $A = \pm(8\mu)^{1/2}$. The first value is now unstable (as we have predicted from the linear analysis), while $A = (8\mu)^{1/2}$ is now stable. This is a classical supercritical bifurcation (cf. Fig. 2).

If we consider modes around the central one, we can obtain the so called *phase winding solutions.* Assuming

$$A = A_K(T)e^{iKX} , \quad (38)$$

we obtain an equation for A_K,

$$A_{KT} - (\mu - K^2)A_K + \tfrac{1}{4}|A_K|^2A_K = 0 . \quad (39)$$

This equation is analogous to (37), but now the bifurcation occurs at $\mu = K^2$ (when the growth rate of the mode crosses the $s = 0$ axis) and the limit amplitude is smaller than A_0.

An analysis of the phase winding solutions shows that these solutions are unstable when $K < (\mu/3)^{1/2}$ (Eckhaus instability). Further analysis introducing the y coordinate would show other types of instabilities, such as the *zig-zag instability.* The interested reader may complete the details along the lines gives in [15].

VII. CONCLUSSION

We know that for values of the Reynolds number below the critical one and in the absence of electric fields, all the modes are stable. However, it is theoretically possible to destabilize the surface with the help of an electric field. We would obtain in this case weakly nonlinear traveling waves with phase velocity, wavelength, and amplitude given by (in nondimensional units)

$$v = 2 , \qquad \lambda = 2\pi\delta\sqrt{\frac{2W}{D}} , \qquad A = \sqrt{\frac{W_eD^{5/2}}{9W^{3/2}} - \frac{2D^3}{9W}} \quad (40)$$

for a value of the electric Weber number slightly above

$$W_e \gtrsim 2\sqrt{DW} \ . \qquad (41)$$

In this case, the instability appears in the form of an unstable band of modes around a central one. It is possible to find an equation of the envelope of this wave packet.

If we look for numerical values of these magnitudes we must take a highly viscous liquid, such as the glycerin. We find that for a layer of about 2 mm, and an angle $\beta \sim 85^\circ$ this destabilization is possible for values of the electric field around 800 kV/m.

ACKNOWLEDGMENT

This work has been supported by the Spanish Dirección General Interministerial de Ciencia y Tecnología under Contract No. PB93-1182.

[1] H.-C. Chang, Ann. Rev. Fluid Mech. **26**, 103 (1994).

[2] S. P. Lin, *Surface Waves in Fluid Interfaces* (Academic Press, New York, 1983), p. 261.

[3] T. B. Benjamin, J. Fluid Mech. **2**, 554 (1957).

[4] C. S. Yih, Phys. Fluids **6**, 321 (1963).

[5] C. Nakaya, Phys. Fluids **18**, 1407 (1975).

[6] S. P. Lin, J. Fluid Mech. **63**, 417 (1974).

[7] A. Pumir, P. Mannevill, and Y. Pomeau, J. Fluid Mech. **135**, 27 (1983).

[8] H.-C. Chang, Chem. Eng. Sci. **42**, 515 (1987).

[9] H.-C. Chang, E. A. Demekhin, and D. I. Kopelevich., J. Fluid Mech. **250**, 433 (1993).

[10] P. L. Kapitza and S. P. Kapitza, Zh. Eksp Teor. Fiz. **19**, 105 (1949).

[11] B. Gjevik, Phys. Fluids **13**, 1918 (1970).

[12] J. Liu, J. D. Paul, and J. P. Gollub, J. Fluid Mech. **250**, 69 (1993).

[13] T. R. Salamon, R. C. Armstrong, and R. A. Brown, Phys. Fluids **6**, 2202 (1994).

[14] D. J. Benney, J. Math. Phys. **45**, 150 (1966).

[15] P. Manneville, *Disipative Structures and Weak Turbulence* (Academic Press, New York, 1990).

[16] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).